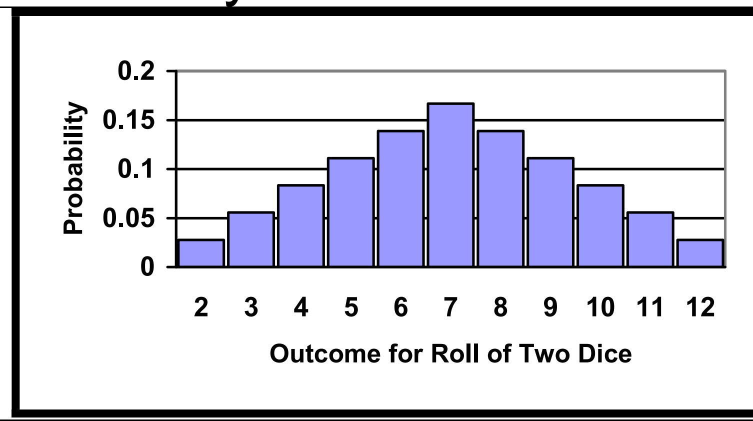
Probabilities and Odds for Games of Chance

Example #11 — A pair of dice is tossed in a game of craps. What is the probability of winning a one roll bet on "any seven?"

There are 6 × 6 = 36 equally likely elementary events:

1 - 1	1 - 2	1 - 3	1 - 4	1 - 5	**1 - 6**
2 - 1	2 - 2	2 - 3	2 - 4	**2 - 5**	2 - 6
3 - 1	3 - 2	3 - 3	**3 - 4**	3 - 5	3 - 6
4 - 1	4 - 2	**4 - 3**	4 - 4	4 - 5	4 - 6
5 - 1	**5 - 2**	5 - 3	5 - 4	5 - 5	5 - 6
6 - 1	6 - 2	6 - 3	6 - 4	6 - 5	6 - 6



6 of the elementary events (in **bold**) are favorable to "any seven"

Probability of "any seven" = ${}^{6}/_{36} = {}^{1}/_{6}$ Odds are "5 to 1 against"

In a fair game, a casino would pay "5 to 1" for "any seven"

However, a casino pays only "4 to 1" House Edge = 1/6 = 16.7%

On average, a player will win one of every six bets on “any seven” — i.e. the odds are “5 to 1 against”

In a fair game, a casino would pay “5 to 1” for “any seven”

							Total
Wager ($):	1	1	1	1	1	1	6
Gain/Loss ($)	-1	-1	-1	-1	-1	+5	0

However, to make money, a casino pays only “4 to 1”

							Total
Wager ($):	1	1	1	1	1	1	6
Gain/Loss ($)	-1	-1	-1	-1	-1	+4	-1

On average, the casino will gain (the player will lose) one dollar for every six dollars bet in playing “any seven”

House edge = $\frac{1}{6} = 16.7\%$

<u>Example #12</u> — A 5-card poker hand is dealt from a shuffled standard deck of 52 cards. What is probability of getting a royal flush (all 5 honor cards "ten, jack, queen, king, ace" in same suit)

Using the Multiplicative Law of Probability for Dependent Events, the probability of getting a royal flush =

Pr (first card is an honor in some suit)

× Pr (second card is an honor in the same suit, given that the first card is an honor in that suit)

× Pr (third card is an honor in the same suit, given that the first two cards are honors in that suit)

× Pr (fourth card is an honor in the same suit, given that the first three cards are honors in that suit)

× Pr (fifth card is an honor in the same suit, given that the first four cards are honors in that suit)

$$= \frac{20}{52} \times \frac{4}{51} \times \frac{3}{50} \times \frac{2}{49} \times \frac{1}{48} = \frac{480}{311{,}875{,}200} = \frac{1}{649{,}740}$$

<u>Example #13</u> — In Blackjack, 2 cards are dealt from a standard deck of 52 cards. What is the probability of getting a "blackjack" — i.e. an ace plus a ten-value card (ten, jack, queen or king)?

There are two different ways of getting a blackjack:

$$\text{Pr (blackjack)} = \text{Pr}\begin{pmatrix}\text{1st card is an ace and 2nd card is a ten-value}\\ \text{OR}\\ \text{1st card is a ten-value and 2nd card is an ace}\end{pmatrix}$$

Using the Multiplicative Rule of Probability for Dependent Events and the Additive Rule of Probability for Events E_1 or E_2:

$$\text{Pr (blackjack)} = \frac{4}{52} \times \frac{16}{51} + \frac{16}{52} \times \frac{4}{51} = \frac{64}{2652} + \frac{64}{2652} = \frac{128}{2652} \cong 0.048$$

Probability of blackjack $\cong \frac{1}{20.7}$ or approximately a "1 in 21 chance"

<u>Example #14</u> — In "The Pick" Arizona Lottery game, you pick six numbers between 1 and 44, trying to match the list of six winning numbers. Cash prizes for 3, 4, 5, 6 correct numbers. What is the probability that you will pick exactly 5 of the winning 6 numbers? (Suppose the winning numbers are: 4 11 15 22 35 38)

Using the Multiplicative Rule of Probability for Dependent Events:

Pr (picking all 6) = $\frac{6}{44} \times \frac{5}{43} \times \frac{4}{42} \times \frac{3}{41} \times \frac{2}{40} \times \frac{1}{39} = \frac{1}{7{,}059{,}052}$

Pr (first 5 numbers you pick correctly match, but 6th does not) =

$$\frac{6}{44} \times \frac{5}{43} \times \frac{4}{42} \times \frac{3}{41} \times \frac{2}{40} \times \frac{38}{39} = \frac{27{,}360}{5{,}082{,}517{,}440}$$

But there are actually six different ways to correctly pick exactly five of the six winning numbers:

✓✓✓✓✓☒ , ✓✓✓✓☒✓ , ✓✓✓☒✓✓ , ✓✓☒✓✓✓ , ✓☒✓✓✓✓ , ☒✓✓✓✓✓

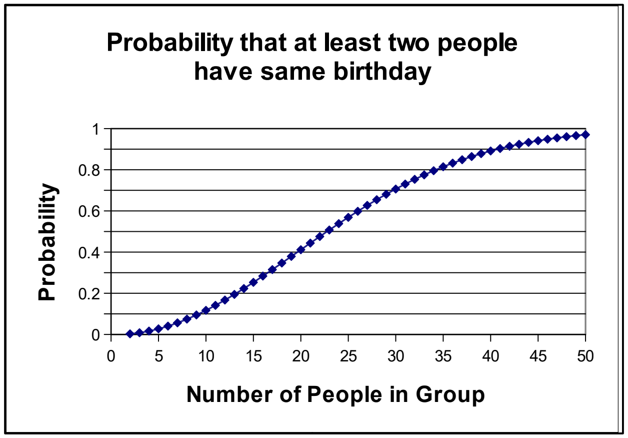
Pr (picking 5 correctly) = $\frac{27{,}360}{5{,}082{,}517{,}440} \times 6 \cong 0.0000323 \cong \frac{1}{30{,}961}$

Numbers and Statistics Can Be Tricky

<u>Example #15</u> — In a random group of 23 people, what is the probability that at least two people share the same birthday?

Group Size	Pr (all different birthdays)	Pr (at least two have same birthday)
2	$\frac{364}{365}$	$1 - \frac{364}{365} \cong 0.003$
3	$\frac{364}{365} \times \frac{363}{365}$	$1 - \frac{364}{365} \times \frac{363}{365} \cong 0.008$
4	$\frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$	$1 - \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \cong 0.016$

Even though 23 is a lot less than 365 days in a year:
In a group of 23 people, the probability that at least two people share the same birthday is 0.507 !



<u>Example #16</u> – Conditional Probabilities Can Be Really Tricky

- For women in their 40s with no symptoms or family history, the probability is 0.8% (8 in a thousand) of having breast cancer
- For women in their 40s with breast cancer, the conditional probability is 90% of having a positive mammogram
- For women in their 40s without breast cancer, the conditional probability is 7% of having a positive mammogram
- What is the conditional probability that a woman in her 40s who has a positive mammogram actually has breast cancer?

- When this question was posed to a group of American doctors, 95 out of 100 guessed the woman's probability of having breast cancer to be somewhere around 75%
- The correct answer is around 9 percent

- This problem can be intuitively worked out by constructing a table of average frequencies

- For women in their 40s with no symptoms or family history, the probability is 0.8% (8 in a thousand) of having breast cancer

	Cancer	No Cancer	TOTAL
Positive Mammogram			
Negative Mammogram			
TOTAL	8	992	1000

- For women with breast cancer, the conditional probability is 90% of having a positive mammogram

	Cancer	No Cancer	TOTAL
Positive Mammogram	7.2		
Negative Mammogram	0.8		
TOTAL	8	992	1000

- For women without breast cancer, the conditional probability is 7% of having a positive mammogram

	Cancer	No Cancer	TOTAL
Positive Mammogram	7.2	69.44	76.64
Negative Mammogram	0.8	922.56	923.36
TOTAL	8	992	1000

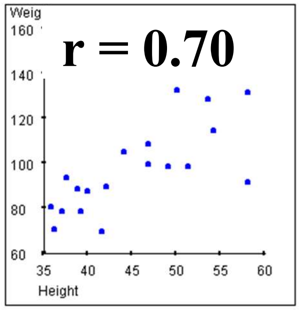
- Pr (Cancer given Pos Mammogram) = ${}^{7.2}/_{76.64} = 9.4\%$
  Pr (Cancer given Neg Mammogram) = ${}^{0.8}/_{923.36} = 0.087\%$

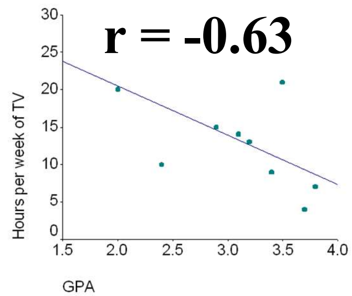
- Despite only 9.4% probability, mammogram test <u>is</u> “splitting risk”
  Pr (cancer given Pos Mam) = 108 × Pr (cancer given Neg Mam)

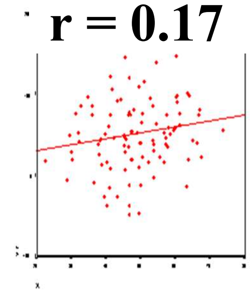
- The reason that the probability of cancer given a positive mammogram is only 9.4% is that the overall rate of cancer for women in their 40s is so low (0.8% or 8 in a thousand)

Correlation Does Not Automatically Imply Causation

Correlation is a measure of the association between two variables







Just because two variables are correlated does not automatically imply that one caused the other. There are 3 main possibilities:

- Variable A may be the cause of variable B, or B the cause of A
- Chance correlation: “Redskins Rule” for president 1932-2000
- Or, it may be that some unknown third variable C is actually the cause of both A and B. This “confounding” variable C can create a spurious relationship between A and B that looks like a causal connection — when in fact there is none.
- I know what you’re thinking: No one will ever get fooled by this. Well, think again. Confounding variables have fooled the best!

Example #17 — A classic example is the strong positive correlation between ice cream sales and drowning incidents. Does eating ice cream cause people to drown at ocean or pool?

- Not really. This is a spurious relationship created by the confounding variable "air temperature," which is associated with both ice cream consumption and drownings in water.

Example #18 — Published in the May 1999 issue of *Nature*, a study at the University of Pennsylvania showed a positive correlation between young children who sleep with the light on and their likelihood of developing myopia in later life

- However, a later study at Ohio State University did not find a causal connection between sleeping with light on and myopia. Instead, it found a strong link between parental myopia and development of child myopia, noting that myopic parents were more likely to leave the light on in their children's bedroom. Aha!
- The confounding variable was parental myopia, which was associated with both child myopia and leaving the light on

<u>Example #19</u> — In a famous example, observational studies in the early 1990s showed that women taking hormone replacement therapy (HRT) had a 40%-50% reduced incidence of coronary heart disease (CHD). By the mid-1990s, HRT became one of the most widely prescribed medications for postmenopausal women.

- To further study the issue, the Women's Health Initiative (which ran between 1993-2005) conducted <u>randomized controlled trials</u> involving more than 16,000 women. Results showed that HRT caused a small, but statistically significant <u>increase</u> in CHD and breast cancer. In fact, the trials were stopped early in 2002 due to a 26% increase in breast cancer. What went wrong and why?
- A re-analysis of the observational data showed that women taking HRT were more likely to be from higher socio-economic groups, with better diet and exercise regimens. Aha!
- The confounding variable was the socio-economic status of the women, which was strongly associated with both HRT and CHD

Moral of the Story

- Observational studies should always be interpreted cautiously, because they sometimes have confounding variables which can create a spurious relationship between two variables that looks like a causal connection — when in fact there is none

- You should be very careful in reading news stories that purport to show that A causes B. News headlines often suggest a causal relationship between two variables, when a closer reading of the article reveals only a correlation.

So, how do you prove a causal relationship?

- Randomized controlled experiments (trials)
- Randomized controlled double-blind clinical trials
- A lot of varied epidemiological evidence (e.g. smoking)
- A scientific understanding of the causal mechanism